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Homotopy Invariants of Time Reversible Periodic Orbits II. Towards Applications

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For a reversible periodic orbit γ we apply the sequence of homotopy invariants $\deg_n(\gamma)$, $n = 1, 2, 3, \ldots$, defined in [Fiedler & Heinze] (1996), We use this sequence to prove a global bifurcation result for reversible periodic orbits with prescribed minimal period. This result will be applied to second order systems with Neumann boundary conditions. A discussion and remarks on the sequence of degrees concludes the paper. © 1996 Academic Press, Inc.

1. INTRODUCTION

We consider time reversible systems of ordinary differential equations, i.e.,

$$\dot{x} = f(x) \qquad x \in \mathbf{R}^{2N}, \qquad f \in C^1, \tag{1.1}$$

with

$$f(Rx) = -f(Rx), \qquad \text{for all} \quad x, \tag{1.2}$$

$$R := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where I denotes the $N \times N$ identity matrix.

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For convenience we repeat some properties of reversible systems. We also give the definition and properties of the sequence of degrees associated to a reversible periodic orbit γ. For further definitions and the proofs, see [Fiedler & Heinze] (1996).

Denote by $\phi_t(x)$ the flow generated by (1.1). A periodic orbit γ is called reversible if it intersects the fixed point space Fix(R) of R in precisely two points p and q. These occur as zeros of the shooting map π_t,

$$\begin{aligned} \pi_t &: \mathbf{R}^N \to \mathbf{R}^N \\ p &\mapsto \varphi_t^-(p), \end{aligned} \tag{1.3}$$

where $p \in \mathrm{Fix}(R)$ and $\varphi_t^-(p)$ indicates the component of $\varphi_t(p)$ in $\mathrm{Fix}(R)^\perp$. If p is a zero of π_t, then so is $q = \varphi_t(p)$. Moreover, $\pi_{nt}(p) = \pi_{nt}(q) = 0$, $n = 1, 2, 3, \ldots$. In the nondegenerate case, we define the sequence of Brouwer degrees $d_n(p)$ and $d_n(q)$ of π_{nt}, at p and q, by

$$d_n(p) = \det D\pi_{nt}(p) \neq 0, \quad d_n(q) = \det D\pi_{nt}(q) \neq 0, \quad n = 1, 2, 3\ldots. \tag{1.4}$$

We also define the orbit degree

$$\deg_n(\gamma) = 1/2(d_n(p) + d_n(q)), \qquad n = 1, 2, 3, \ldots. \tag{1.5}$$

These degrees are related via the complex Floquet multipliers on the unit circle $e^{i\pi\vartheta_j}$, $j = 1, \ldots, r$, and σ_-, the number of real Floquet multipliers in $(-\infty, -1)$. This is expressed in Theorem 2.1 in [Fiedler & Heinze] (1996), which we recall next.

1.1. THEOREM. *Let γ be a nondegenerate reversible periodic orbit. Let n be any positive integer. Then*

$$d_{2n-1}(p) = d_1(p) \cdot (-1)^{(n-1)\sigma_-} \cdot \prod_{j=1}^{r} (-1)^{[(n-1/2)\vartheta_j]}, \tag{1.6a}$$

$$d_{2n}(p) = d_2(p) \cdot (-1)^{(n-1)\sigma_-} \cdot \prod_{j=1}^{r} (-1)^{[n\vartheta_j]}, \tag{1.6b}$$

where $[\cdot]$ denotes the integer part. The same holds for q replacing p. If $r = 0$, that is for hyperbolic γ, we assign the usual value 1 to the empty products over j. The degrees at p and q are related:

$$d_n(p) = d_n(q) \qquad \textit{for odd } n, \tag{1.7a}$$

$$d_n(p) = (-1)^{\sigma_-} d_n(q) \qquad \textit{for even } n. \tag{1.7b}$$

In particular,

$$\deg_{2n-1}(\gamma) = \deg_1(\gamma) \cdot (-1)^{(n-1)\sigma_-} \prod_{j=1}^{r} (-1)^{[(n-1/2)\vartheta_j]}, \tag{1.8a}$$

$$\deg_{2n}(\gamma) = \deg_2(\gamma) \cdot (-1)^{(n-1)\sigma_-} \prod_{j=1}^{r} (-1)^{[n\vartheta_j]}, \tag{1.8b}$$

where $\deg_{2n}(\gamma) = 0$ *for odd* σ_- *and all* n.

These formulas will be used in Section 2 to prove a global bifurcation result. In Section 3 we apply this result to second order Neumann problems. We also recall some terminology.

1.2. Definition. Let γ be a reversible periodic orbit. Assume that -1 is not a Floquet multiplier, and $+1$ is only a trivial multiplier: algebraically double and geometrically simple.

We call γ *elliptic*, if there are nontrivial Floquet multipliers on the unit circle. Otherwise, we call γ *hyperbolic*. Let σ_- count the real Floquet multipliers in $(-\infty, -1)$. We call γ *Möbius*, if σ_- is odd. Otherwise we call γ *non-Möbius*. We call $\sigma_-(\text{mod } 2)$ the *Möbius parity* of γ.

2. A Global Bifurcation Theorem

In this section we will prove the following global bifurcation result for reversible periodic orbits.

2.1. Theorem. *Let the following three assumptions hold for the reversible system* $\dot{x} = f(x)$ *and some* $T > 2$.

(i) *Among the reversible equilibria* $\xi \in \text{Fix}(R)$ *there is exactly one, say* $\xi = 0$, *with eigenvalues on the imaginary axis. These are the simple eigenvalues* $\pm\pi i$.

(ii) *The set of reversible periodic orbits with period* $\leqslant T$ *is bounded.*

(iii) *All reversible periodic orbits of (not necessarily minimal) period* T *are nondegenerate.*

Then there exists a nondegenerate reversible periodic orbit γ *of period* T *such that one of the following three mutually exclusive statements* $(a-c)$ *holds.*

(a) γ *is hyperbolic non-Möbius with minimal period* T.

(b) γ *is hyperbolic Möbius with minimal period* T *or* $T/2$.

(c) γ *is elliptic, but* T *might not be its minimal period.*

More precisely, assume that all reversible periodic orbits of (not necessarily minimal) period T are hyperbolic. For $t>0$ let a_t, (c_t), respectively, denote the number of non-Möbius, (Möbius) reversible periodic orbits of minimal period t. Then

$$a_T + c_T + c_{T/2} \text{ is odd.} \tag{2.1}$$

In a way, this theorem generalizes the case $N=1$. Indeed, neither Möbius nor elliptic orbits can occur in two dimensions. Only case (a) remains. Unlike usual results on global Hopf bifurcation, however, our theorem does not assert the reversible orbits to form a continuum or a connected curve.

2.2. Lemma. *Let $x=\xi \in \mathrm{Fix}(R)$ be a reversible equilibrium of a reversible system $x=f(x)$, that is, $\pi_t(\xi)=0$ for all t. Assume that the linearization, the block matrix $M=f'(\xi)=\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, is nonsingular and*

$$\dot{y} = My \tag{2.2}$$

does not possess a solution of period $2t$. Then the local contribution of ξ to $\deg \pi_t$ *is given by*

$$\deg_{\mathrm{loc}}(\pi_t, \xi) := \operatorname{sign} \det \pi_t'(\xi) = \operatorname{sign} \det C \cdot \prod_{\substack{i\omega \in \mathrm{spec}(M) \\ \omega > 0}} \operatorname{sign} \sin(\omega t). \tag{2.3}$$

In the product, multiple eigenvalues are repeated with algebraic multiplicity; an empty product equals 1.

Proof. By perturbation invariance we may assume all eigenvalues of M to be simple. By reversibility, $R\xi=\xi$ implies $MR=-RM$. Hence $A=D=0$. Moreover, $\kappa \in \mathrm{spec}(M)$ if and only if

$$\kappa^2 \in \mathrm{spec}(BC) = \mathrm{spec}(CB). \tag{2.4}$$

Indeed, by some row manipulations,

$$\det \begin{pmatrix} -\kappa & B \\ C & -\kappa \end{pmatrix} = \det \begin{pmatrix} -\kappa + \kappa^{-1} BC & 0 \\ C & -\kappa \end{pmatrix} = \det(\kappa^2 - BC),$$

for the nontrivial case $\kappa \neq 0$. In particular, $i\omega \in \mathrm{spec}(M)$, $\omega>0$ generate precisely the negative eigenvalues $-\omega^2$ of BC. Now exponentiate M, letting

$$\begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix} = \exp(Mt), \tag{2.5}$$

for $t > 0$. Note that

$$\deg_{\mathrm{loc}}(\pi_t, \xi) = \operatorname{sign} \det C_t, \tag{2.6}$$

by definition. We claim

$$C_t = \left(\sum_{j=0}^{\infty} \frac{t^{2j+1}}{(2j+1)!} (CB)^j \right) C. \tag{2.7}$$

Indeed, an easy induction shows that

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}^{2j} = \begin{pmatrix} (BC)^j & 0 \\ 0 & (CB)^j \end{pmatrix},$$

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}^{2j+1} = \begin{pmatrix} 0 & (BC)^j B \\ (CB)^j C & 0 \end{pmatrix},$$

for all integer $j \geqslant 0$. Plugging this into the exponential series for $\exp(Mt)$ proves (2.7). By (2.6), (2.7), and (2.4)

$$\begin{aligned} \deg_{\mathrm{loc}}(\pi_t, \xi) &= \operatorname{sign} \det C \cdot \operatorname{sign} \prod_{\tilde{\kappa} \in \operatorname{spec}(BC)} \left(\sum_{j=0}^{\infty} \tilde{\kappa}^j \frac{t^{2j+1}}{(2j+1)!} \right) \\ &= \operatorname{sign} \det C \cdot \operatorname{sign} \prod_{\substack{\tilde{\kappa} \in \operatorname{spec}(BC) \\ \tilde{\kappa} < 0}} \left(\sum_{j=0}^{\infty} \tilde{\kappa}^j \frac{t^{2j+1}}{(2j+1)!} \right) \\ &= \operatorname{sign} \det C \cdot \prod_{\substack{i\omega \in \operatorname{spec}(M) \\ \omega > 0}} \operatorname{sign} \left(\frac{1}{\omega} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} (\omega t)^{2j+1} \right) \\ &= \operatorname{sign} \det C \cdot \prod_{\substack{i\omega \in \operatorname{spec}(M) \\ \omega > 0}} \operatorname{sign} \sin(\omega t). \end{aligned}$$

This proves the lemma. ▮

We note that Lemma 2.2 could also be derived from Theorem 1.1, essentially putting $p = q = \xi$.

With Lemma 2.2 and under assumptions (i)–(iii) in Theorem 2.1 we can now compute the contribution of the reversible periodic orbits to the degree $\deg(\pi_t)$ at positive non-integer $t \leqslant T/2$. Let $E \subseteq \operatorname{Fix}(R)$ denote the set of reversible equilibria. Consider the uniformly bounded sets

$$P_t := \{ x \in \operatorname{Fix}(R) \mid \pi_t(x) = 0 \} \setminus E \tag{2.8}$$

of nonstationary points with period $2t$. In Lemma 2.3 below we prove that the set P_t is in fact compact. To compute the contribution of P_t to $\deg(\pi_t)$, for $t \leqslant T/2$, we choose a large ball $B \subseteq \mathrm{Fix}(R)$ containing all intersections with $\mathrm{Fix}(R)$ of reversible periodic orbits with period $t \leqslant T/2$, according to assumption 2.1(ii). Moreover, we assume that no reversible equilibria lie on the boundary of B. Let

$$P_t^0 \subseteq B \backslash E \tag{2.9}$$

denote an open neighborhood of the compact set P_t of reversible periodic "orbits." Choose $P_t^0 = \varnothing$ if $P_t = \varnothing$. We will compute $\deg(\pi_t, P_t^0, 0)$ in two different ways. First we use homotopy invariance; see (2.10) in Lemma 2.3 below. Second we add, mod 2, the individual contributions of reversible periodic orbits of "period" $2t$; see (2.16). Comparing will prove Theorem 2.1.

Global existence of φ_t and π_t will be a minor technical point later on. Since the relevant reversible periodic orbits are in some large bounded set, we may multiply the vector field outside by some positive scalar function, symmetric with respect to R, such that the maximal local flow φ_t becomes a global flow. Orbits are unchanged by this modification. Thus we will only consider global flows below.

2.3. LEMMA. *Let $0 < t \leqslant T/2$ be non-integer. Then the set P_t of reversible periodic orbits of period $2t$ defined in* (2.8) *is compact. The Brouwer degree of π_t in its open neighborhood P_t^0 is given by*

$$\deg(\pi_t, P_t^0, 0) = e_0(1 - \mathrm{sign}\,\sin(\pi t)), \tag{2.10}$$

for some constant $e_0 \in \{\pm 1\}$.

Proof. We prove compactness of the set P_t first. The set P_t is bounded, by assumption 2.1(ii). It is closed by assumption 2.1(i) as we now prove, indirectly. If a sequence of non-stationary periodic orbits with period $2t$ converges to a reversible equilibrium ξ, then the linearization (2.2) at ξ must also possess a nontrivial periodic solution with period $2t$. This fact is a consequence of the virtual period proposition; see e.g. [Mallet–Paret & Yorke] (1982), Proposition 3.1. In particular, the linearization at ξ must possess purely imaginary eigenvalues. In our case, $\xi = 0$ by assumption 2.1(i), and therefore $t \in \mathbf{Z}$, again by 2.1(i). This contradicts our assumption $t \notin \mathbf{Z}$. Therefore, the set P_t is indeed compact.

We compute $\deg(\pi_t, P_t^0, 0)$ next. By additivity of the Brouwer degree,

$$\deg(\pi_t, P_t^0, 0) = \deg(\pi_t, B, 0) - \sum_{\xi \in E \cap B} \deg_{\mathrm{loc}}(\pi_t, \xi). \tag{2.11}$$

We split the sum over $\xi \in E \cap B$ into the contribution from the reversible Hopf point $\xi = 0$ and a remainder, e, from reversible hyperbolic equilibria:

$$\sum_{\xi \in E \cap B} \deg_{\text{loc}}(\pi_t, \xi) = \deg_{\text{loc}}(\pi_t, \xi = 0) + e. \tag{2.12}$$

Both contributions are known from Lemma 2.2. Note that e does not depend on $t > 0$. In contrast,

$$\deg_{\text{loc}}(\pi_t, \xi = 0) = e_0 \cdot \text{sign} \sin(\pi t), \tag{2.13}$$

for some fixed $e_0 = \pm 1$.

To simplify the right hand side of (2.12), let $\tau_{\min} < 1$ be chosen small enough such that

$$P_t = \varnothing, \qquad \text{for} \quad t \leqslant \tau_{\min}. \tag{2.14}$$

By boundedness assumption 2.1(ii) such a lower bound $2\tau_{\min}$ on the periods exists; see [Lasota & Yorke] (1971). For such t, (2.11)–(2.12) imply

$$\deg(\pi_t, B, 0) = e + e_0, \tag{2.15}$$

since $t \leqslant \tau_{\min} < 1$ and $P_t^0 = \varnothing$. By homotopy invariance of degree, (2.15) holds for $0 < t \leqslant T/2$. Thus we may rewrite (2.11)–(2.12) for general non-integer $0 < t \leqslant T/2$ as

$$\deg(\pi_t, P_t^0, 0) = e_0(1 - \text{sign} \sin(\pi t)).$$

This proves (2.10), and the lemma. ∎

So much for the homotopy way to compute $\deg(\pi_t, P_t^0, 0)$. Next, we compute the same quantity by adding up the local contributions of all reversible periodic orbits with period $2t = T$. By compactness of P_t and by our nondegeneracy assumption 2.1(iii) there are only finitely many such orbits. Recall that we assume them not to be of elliptic type. Because $T = 2t$ may not be the minimal period, being just a period, we have to account for minimal periods T/j, $j = 1, 2, 3, \ldots$ and sum over their respective contributions. To do this denote

$a_{T/j}$: the number of hyperbolic non-Möbius reversible periodic orbits with minimal period T/j,

$c_{T/j}$: the number of hyperbolic Möbius reversible periodic orbits with minimal period T/j,

as in Theorem 2.1. Note that $a_{T/j}$, $c_{T/j}$, $j = 1, 2, 3, \ldots$ count *all* orbits with period T, in absence of elliptic orbits. Adding their mod 2 contributions to $(1/2)\deg(\pi_t, P_t^0, 0)$, for $0<t=T/2\notin \mathbf{Z}$, we claim

$$\sum_j a_{T/j} + \sum_{j\ \text{odd}} c_{T/j} \equiv [T/2] (\bmod 2). \tag{2.16}$$

The sums are running over integer $j\geqslant 1$, with the specified restrictions. Note that $a_{T/j}=c_{T/j}=0$ for $T/j\leqslant 2\tau_{\min}$, by the lower period bound (2.14). Therefore the sums are finite. As usual $[t]$ denotes the largest integer $\leqslant t$.

To prove (2.16), we invoke Lemma 2.3, and Theorem 1.1. By (2.10) at $t=T/2$,

$$\tfrac{1}{2}\deg(\pi_t, P_t^0, 0) = \pm(1-\operatorname{sign}\sin(\pi t))/2 \equiv [t] (\bmod 2) \tag{2.17}$$

equals the right hand side of (2.16).

Consider a hyperbolic non-Möbius orbit γ of minimal period $T/j=2\tau$ next. By Theorem 1.1 it contributes

$$\deg_j(\gamma) = \pm 1 \equiv +1 (\bmod 2) \tag{2.18}$$

to $(1/2)\deg(\pi_t, P_t^0, 0)$, $t=2j\tau$. Indeed, σ_- is even for a non-Möbius orbit, $d_1(p)$ and $d_2(p)$ are ± 1, and $r=0$ for a hyperbolic orbit. This explains the sum over $a_{T/j}$ in (2.16). Consider a hyperbolic Möbius orbit γ of minimal period $T/j=2\tau$ next. Since σ_- is odd, this time, it contributes (2.18) to $(1/2)\deg(\pi_t, P_t^0, 0)$ if and only if j is odd. Since "elliptic" orbits do not occur, additivity of Brouwer degree thus proves (2.16).

For later use, we need a variant of (2.16). In fact (2.16) holds for any fraction T/k, $k=1, 2, 3, \ldots$ replacing T. Thus

$$\sum_j a_{T/kj} + \sum_{j\ \text{odd}} c_{T/kj} \equiv [T/(2k)] (\bmod 2), \qquad \text{for all} \quad k\geqslant 1. \tag{2.19}$$

2.4. Lemma. *Let $a_{T/j}$, $c_{T/j}$, $j\geqslant 1$, be any sequence of integers satisfying (2.19) for all $k\geqslant 1$. Assume $a_{T/j}=c_{T/j}=0$ for all $j>j_{\max}$. Then*

$$a_{T/k}+c_{T/k}+c_{T/(2k)} \equiv \begin{cases} 1\ (\bmod 2), & \text{for} \quad k\leqslant T/2 \\ 0\ (\bmod 2), & \text{for} \quad k> T/2 \end{cases} \tag{2.20}$$

holds for all integers $k\geqslant 1$.

Proof. We recast (2.19) in a more convenient form, summing over all $k\geqslant 1$. Of course, all sums will in fact be finite. For integers, let $j\mid m$ indicate that j divides m; by $D(m)$ we denote the number of divisors j of m, including $j=1$ and $j=m$. Note that

$$D(m) = (\eta_1+1)\cdots(\eta_\mu+1), \tag{2.21}$$

if $m = p_1^{\eta_1} \cdots p_\mu^{\eta_\mu}$ is the prime factor decomposition. In particular

$$D(m) \equiv 1 \pmod 2 \tag{2.22}$$

if m is a square. Otherwise $D(m) \equiv 0$. For any integer m, let m_{odd} denote the maximal odd factor of m. Throughout, let $\equiv$ denote equality mod 2. With these conventions we recast (2.19), substituting $m = kj$

$$\begin{aligned}
\sum_k [T/(2k)] &\equiv \sum_k \sum_j a_{T/kj} + \sum_k \sum_{j\,\text{odd}} c_{T/kj} \\
&\equiv \sum_m \Big(\sum_{j \mid m} 1 \Big) a_{T/m} + \sum_m \Big(\sum_{j \mid m_{\text{odd}}} 1 \Big) c_{T/m} \\
&\equiv \sum_m D(m)\, a_{T/m} + \sum_m D(m_{\text{odd}})\, c_{T/m} \\
&\equiv \sum_m a_{T/m^2} + \sum_{\alpha \geqslant 0} \sum_{m\,\text{odd}} c_{T/(2^\alpha m^2)} \\
&\equiv \sum_{\alpha \geqslant 0,\, \alpha\ \text{even}} \sum_{m\,\text{odd}} (a_{T/(2^\alpha m^2)} + c_{T/(2^\alpha m^2)} + c_{T/(2 \cdot 2^\alpha m^2)}).
\end{aligned} \tag{2.23}$$

We have used (2.22) and replaced m by m^2. All sums run over all indices $\geqslant 1$, unless otherwise indicated. Again (2.23) holds true for T replaced by $T' = T/k'$, $k' = 1, 2, 3, \ldots$. The form (2.23), containing expressions

$$a_{T'} + c_{T'} + c_{T'/2} \tag{2.24}$$

for certain fractions $T' = T/k'$ on the right, lends itself to an inductive proof over increasing $T' = T/k'$.

To begin the induction, consider $T' = T/k' < T/j_{\max}$. Then the terms (2.24) are all zero. Recursively, we increase T' but keep $T'/2 < 1$. Since the left hand side of (2.23) remains zero, we conclude

$$a_{T/k} + c_{T/k} + c_{T/(2k)} \equiv 0 \tag{2.25a}$$

as long as

$$T/k < 2. \tag{2.25b}$$

This proves the bottom part of (2.20).

To begin the induction over increasing T' for the top part of (2.20), let $T' = T/k'$ be chosen minimal such that $T'/2 \geqslant 1$. Invoking (2.23), with T' replacing T, and (2.25a) yields

$$a_{T'} + c_{T'} + c_{T'/2} \equiv 1. \tag{2.26a}$$

For general $T'/2 \geqslant 1$ note that the right hand side of (2.23) contains precisely $[\sqrt{T/2}]$ terms (2.24) with $(1/2)\,T' = (1/2)\,T/(2^{\alpha} m^2) \geqslant 1$. Therefore we will have proved (2.26a) for all

$$T' = T/k \geqslant 2, \tag{2.26b}$$

recursively, if we only prove the following claim

$$\sum_k [t/k] \equiv [\sqrt{t}] \pmod 2, \qquad \text{for all} \quad t = T/2 > 0. \tag{2.27}$$

Note that (2.26a, b) prove the top part of (2.20). It remains to prove (2.27). This relation was observed by [Dirichlet] (1887), p. 52; for a proof see also [Pólya & Szegö] (1976), exercises VIII. 79, 80. We reproduce the short argument. In planar cartesian (x_1, x_2)-coordinates, the left hand side of (2.27) counts the integer lattice points in the set

$$\{(x_1, x_2) \mid 0 < x_1, x_2, \text{ and } x_1 x_2 \leqslant t\}, \tag{2.28}$$

adding up the vertical slices $x_1 = k$, $0 < x_2 \leqslant [t/x_1]$. Since the set (2.28) and the integer lattice are invariant under a reflection at the diagonal $x_1 = x_2$, only the diagonal of (2.28) contributes, mod 2. On $x_1 = x_2$ the set (2.28) obviously contains exactly $[\sqrt{t}]$ lattice points, which is the right hand side of (2.27). The lemma is therefore proved. ∎

With Lemma 2.4 the proof of Theorem 2.1 is complete, provided $t = T/2 > 1$ is not an integer. We now treat the remaining case $2 \leqslant t = T/2 \in \mathbf{Z}$. The sum

$$a_T + c_T + c_{T/2} \tag{2.29}$$

accounts for nondegenerate hyperbolic non-Möbius reversible periodic orbits of minimal period $T \geqslant 4$, and also for such orbits with minimal period $T/2 \geqslant 2$ of Möbius type. By local reversible Hopf bifurcation such orbits cannot accumulate at any equilibrium ξ. Indeed, the minimal periods near the only candidate, the reversible Hopf point $\xi = 0$, are located near the limiting value 2; see e.g. [Vanderbauwhede] (1982). Moreover, the reversible periodic orbits bifurcating from $\xi = 0$ are not of Möbius type. In particular, the sum (2.29) is finite and remains invariant under small perturbations $T - \varepsilon$ of T. Since $k(T-\varepsilon)/2 \notin \mathbf{Z}$ can be assumed, the above results imply

$$a_T + c_T + c_{T/2} = a_{T-\varepsilon} + c_{T-\varepsilon} + c_{(T-\varepsilon)/2} \equiv 1 \pmod 2, \tag{2.30}$$

and Theorem 2.1 is proved. ∎

3. Discussion

In this section we explore some largely open neighborhood of our results. We begin with the question of homotopy invariance of the whole sequence $\deg_n(\gamma)$, $n = 1, 2, 3, ...,$ and related objects. This question is closely tied to subharmonic bifurcation. We then get entangled in the notorious similarities between Hamiltonian and reversible systems. Since our result on periodic orbits with prescribed minimal period is quite different in spirit, we also include some remarks on global Hopf bifurcation which aims at unbounded continua of periodic orbits. After a brief excursion into partial differential equations, we finally summarize our results for second order boundary value problems.

To study homotopy invariance of our orbit degree $\deg_n(\gamma)$ let γ_0 be a nondegenerate reversible periodic orbit, with associated p_0, q_0 and minimal half period τ_0 as in Section 1.2 of [Fiedler & Heinze] (1996). Since

$$\pi_{\tau_0}(p_0) = 0, \qquad \det \pi'_{\tau_0}(p_0) \neq 0, \tag{3.1}$$

the orbit γ_0 continues to a local branch $\gamma(\tau)$, through $p(\tau)$ nearby $p(\tau_0) = p_0$. If we also assume γ_0 to be hyperbolic, then we know that

$$\deg_n(\gamma(\tau)) = \deg_n(\gamma_0) = \begin{cases} (-1)^{(n-1)\sigma_-/2} \deg_1(\gamma_0) & \text{for odd } n, \\ \deg_2(\gamma_0) & \text{for even } n, \end{cases}$$

holds. In particular, the $\deg_n(\gamma(\tau))$ are independent of τ in a neighborhood of τ_0 which can be chosen independently of n.

In contrast, suppose next γ_0 is elliptic. Then the $\deg_n(\gamma(\tau))$ will typically oscillate rapidly as τ varies, especially for large n. Specifically, let $\exp(\pm \pi i \vartheta_j(\tau))$ denote pairs of simple Floquet multipliers on the unit circle and suppose that one of them crosses a primitive nth root of unity as τ varies. For simplicity, also assume σ_- is even. Then $\deg_{kn}(\gamma(\tau))$, $k = 1, 2, 3, ...,$ change sign, accordingly, since $\sin((1/2)\, kn\, \pi \vartheta_j(\tau))$ does. By standard degree theory, this implies that additional zeros of π_t bifurcate from the primary branches $(n\tau, p(\tau))$ or $(n\tau, q(\tau))$. The associated bifurcation is a subharmonic bifurcation; see e.g. [Vanderbauwhede] (1990) and the references there. In Fig. 3.1a–3.1d we schematically depict a subharmonic bifurcation, together with a few other cases. The vertical axis indicates Fix(R). The horizontal axis may be viewed as "time" τ; but note that we identify points on the same orbit and omit the multiples of the minimal τ. Also, the secondary $n\tau$-branch is drawn near the primary branch to emphasize the bifurcation aspect. In fact, Fig. 3.1d ignores certain details of the subharmonic bifurcation. For example, suppose $n \geqslant 4$ is even. Then there are in fact two secondary half-branches, one terminating at p and the other at q.

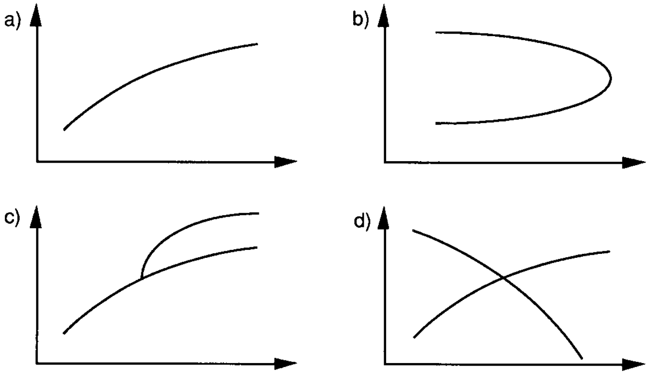

FIG. 3.1. Bifurcations of periodic orbits with minimal period near T. (a) hyperbolic, no bifurcation, (b) saddle node, (c) period doubling, (d) subharmonic, $n \geqslant 3$.

Since the set of nth roots of unity, $n = 3, 4, 5, \ldots$ is dense on the unit circle, we also expect the set of subharmonic bifurcation points to be dense, typically, along any elliptic branch. It was observed by [Vanderbauwhede] (1990), p. 956, that also one of the secondary branches might be elliptic. And so on, for a *cascade of subharmonic bifurcations*. We want the alternative in Theorem 2.1, minimal period versus elliptic orbit, to be understood in this framework.

By our definition, an elliptic orbit is allowed to also possess Floquet multipliers off the unit circle. Only in two degrees of freedom, $x \in \mathbf{R}^{2N}$, $N = 2$, does ellipticity imply linearized *stability*. Typically, in fact, nonlinear stability also holds due to the existence of families of two-dimensional KAM tori; see [Sevryuk] (1986). For $N = 1$, clearly, Möbius orbits and elliptic orbits are equally impossible. But that case is amenable to phase plane analysis anyhow. See e.g. [Schaaf] (1990) for a recent study.

We now sketch an idea on how to drop the *nondegeneracy assumption* (iii) in Theorem 2.1. In fact, by genericity arguments in the spirit of [Mallet-Paret & Yorke] (1982), [Fiedler] (1985) and [Fiedler] (1988), it is possible to find a sequence $f_m \to f$ of reversible systems for which assumptions (i)–(iii) and, therefore, conclusions (a)–(c) hold. (Admittedly, we omit some details here.) In the limit $m \to \infty$, however, minimal periods may drop by an integer factor $n \geqslant 2$, similarly as for subharmonic bifurcations. In that case, the limiting periodic orbit γ must possess a Floquet multiplier χ which is an n th root of unity, by the virtual period proposition; see e.g. [Chow *et al.*] (1983), [Fiedler] (1985). Therefore, conclusions (a)–(c) remain valid if we also allow "elliptic" orbits to just possess Floquet multipliers -1. Our oddness claim (2.1), however, has to be dropped since finiteness of $a_T + c_T + c_{T/2}$ cannot be guaranteed in degenerate cases.

It is tempting to recast the infinite sequence $\deg_n(\gamma)$ of degrees into a more analytic quantity. *Zeta functions* are one possibility. Given a sequence $\underline{d} = (d_1, \ldots)$ of integers, let

$$\zeta^{\underline{d}}(s) := \sum_{n=1}^{\infty} d_n n^{-s}. \tag{3.3}$$

The sum converges absolutely for Re $s > \beta + 1$, provided

$$d_n \leqslant C n^{\beta} \tag{3.4}$$

for some constants C, β. For example, we may consider

$$d_n = \tfrac{1}{2} \deg(\pi_{nt}, P_{nt}, 0) \in \{-1, 0, 1\}, \tag{3.5}$$

for the set P_{nt} of reversible periodic orbits defined as in (2.8); see also Lemma 2.3. Which reversible periodic orbits γ, τ appear in P_{nt}? Certainly τ and t must be rationally related, that is

$$l\tau = jt \tag{3.6}$$

for some relatively prime positive integers l, j. Moreover

$$n = kj \tag{3.7}$$

must be a multiple of j. Let

$$\sigma_k = \deg_{kl}(\gamma) \tag{3.8}$$

and call $\underline{\sigma} = (\sigma_1, \sigma_2, \ldots)$ the *type* of the nondegenerate orbit γ. For fixed $\underline{\sigma}$ let $a_j(\underline{\sigma})$ denote the number of reversible periodic orbits γ of type $\underline{\sigma}$ for which (3.6) holds. Then (3.5)–(3.8) with the notation (3.3) imply

$$\sum_{\underline{\sigma}} \zeta^{\underline{a}(\underline{\sigma})} \cdot \zeta^{\underline{\sigma}} = \zeta^{\underline{d}}. \tag{3.9}$$

Here $\underline{a}(\underline{\sigma}) = (a_1(\underline{\sigma}), a_2(\underline{\sigma}), \ldots)$; we assume that only a finite number of types occurs, and we assume convergence as in (3.4) for $\underline{a}(\underline{\sigma})$. To prove (3.9) just note that by definition

$$\sum_{\underline{\sigma}} \sum_{j \mid n} a_j(\underline{\sigma})\, \sigma_{n/j} = d_n. \tag{3.10}$$

For illustration, suppose that only the constant non-Möbius hyperbolic types $\underline{\sigma} = \pm(1, 1, 1, 1, \ldots) = \pm \underline{e}$ are present. Defining $a_j := a_j(\underline{e}) - a_j(-\underline{e})$ we obtain $\sum_{j \mid n} a_j = d_n$.

By the Möbius inversion theorem,

$$a_n = \sum_{j \mid n} \mu(j)\, d_{n/j} \tag{3.11}$$

where $\mu(j)$ is the Möbius function: $\mu(j) = (-1)^l$ if j is a product of l distinct primes, and $\mu(j) = 0$ otherwise. See e.g. [Abramowitz & Stegun] (1965). Note that

$$|a_n| \leqslant 2^l \cdot \sup_n |d_n| \tag{3.12}$$

if n contains l distinct prime factors. Therefore, $|a_n|$ grows very slowly for bounded sequences d_n.

Still, the rather restrictive growth condition (3.4) impedes our definition (3.3) of the zeta function $\zeta^{\underline{d}}$, $\zeta^{\underline{a}}$. Other definitions are possible, for example

$$Z^{\underline{d}}(x) = \sum_{n=1}^{\infty} d_n x^{n-1}, \tag{3.13a}$$

or more generally

$$Z^{\underline{d}}_{\underline{\varphi}}(x) = \sum_{n=1}^{\infty} d_n \varphi_n(x) \tag{3.13b}$$

for a linearly independent system $\varphi_n(x)$ of functions.

Let a_n denote the number of reversible periodic orbits of period n. Then the radius of convergence s of $Z^{\underline{d}}$ satisfies $\log(1/\rho) \leqslant \limsup((1/n) \log a_n)$ allowing for an exponential rather than just polynomial growth of a_n. Also note that $Z^{\underline{d}}$ is related to the logarithmic derivative of a "*false zeta function*" in the spirit of [Smale] (1967). Unfortunately, the coupling of local degrees of reversible periodic orbits with rationally related periods is not as easily expressed, in this case, as in (3.9).

In Theorem 2.2 in [Fiedler & Heinze] (1996) we recovered information on the Floquet multipliers from the sequence of degrees. This suggests yet another approach to the problem of recasting the whole sequence $\deg_n(\gamma)$ into a manageable form. Indeed, we could associate to γ a *spectral degree* given by explicit expressions for the Fourier coefficients a_λ of the series $\deg_n(\gamma)$. The map from the $\deg_n(\gamma)$ to the a_λ is linear and one-to-one. In particular, additivity of degree carries over to the a_λ. One can therefore base an analysis on the a_λ instead of the $\deg_n(\gamma)$.

We do not pursue these questions any further, here. Instead, note the difference between our present global result and the approach which we took in Theorem 2.1. Rather than considering the infinitely many multiples nT of a period T we took effectively a finite number of integers fractions T/n, that is, all multiples of the basic "concert pitch" frequency $1/T$: harmonics instead of subharmonics, following an ancient tradition in musicology.

We have lamented above about the apparent lack of continuity in the periodic solutions found. Results on *global Hopf bifurcation* typically address this point. See for example [Alexander & Yorke] (1978), [Geba & Marzantowicz], [Ize *et al.*] (1989), and the references there. One popular approach is the following. Rescaling period T to one, we may seek 1-periodic solutions $x(\cdot)$ of

$$F(T, x) := -\frac{1}{T}\dot{x} + f(x) = 0. \tag{3.14}$$

Note that F commutes with the group $SO(2) = \mathbf{R}/\mathbf{Z}$ acting by time shift:

$$(\theta x)(t) := x(t+\theta), \qquad \theta \in \mathbf{R}/\mathbf{Z}. \tag{3.15}$$

Indeed, f is autonomous. The parameter $(1/T)$ multiplies the infinitesimal generator (d/dt) of the group action. If f is reversible, then $O(2)$, generated by the above $SO(2)$ and a reflection κ, commutes with F. Here κ acts by

$$(\kappa x)(t) : = Rx(-t). \tag{3.16}$$

Inserting an additional real parameter λ, unbounded continua of periodic solutions can be obtained by abstract topological methods in the $SO(2)$ case, under suitable assumptions; see the above references. The $O(2)$ case can be treated similarly; note that reversible periodic orbits are fixed under κ. In all their abstract elegance and beauty, these results tend to ignore a quite special property of periodic solutions $x(\cdot)$ which we have emphasized very much: if $(T, x(t))$ is a 1-periodic solution then so is $(nT, x(nt))$, for any $n \geqslant 1$. Such a rescaling property is absent for general $SO(2)$- or $O(2)$-equivariant problems. It is correspondingly difficult to control the minimal period alias, in group jargon, the isotropy of $x(\cdot)$ in the resulting unbounded continua.

"*Snakes*" are a remedy, at least in the generic case of non-reversible vector fields f. See the original paper [Mallet-Paret & Yorke] (1982), and [Fiedler] (1988) for detailed references. As we recall from the introduction, the basic tool is an orbit index $\phi(\gamma)$ which averages the local fixed point indices $i_n = i(\Pi^n)$, $n \geqslant 1$, of the iterates of a Poincaré map Π of γ. In fact

$$\phi(\gamma) = \tfrac{1}{2}((-1)^{\sigma_+} + (-1)^{\sigma_+ + \sigma_-}), \tag{3.17}$$

in the nondegenerate case where σ_+ counts the real Floquet multipliers in $(1, \infty)$ and $(-1)^{\sigma_-}$ is essentially the Möbius parity. We claim that $(-1)^{\sigma_+}$, $(-1)^{\sigma_-}$, and therefore all $i(\Pi^n)$, are determined by our orbit degree deg (γ), in the reversible case, if we ignore for a moment the fact that periodic orbits are not isolated. Indeed the trivial Floquet multiplier $+1$ may be assumed to have algebraic multiplicity two. It is easy to see, that, if χ is a Floquet multiplier, for a reversible system, then so are $\bar{\chi}$, χ^{-1} and $\bar{\chi}^{-1}$. This implies

$$\sigma_+ + \sigma_- + r + 1 \equiv N \pmod 2,$$

where r counts the "elliptic" pairs. In [Fiedler & Heinze] (1996), Theorem 2.2, we have computed r and σ^- mod 2 from deg (γ). This proves our claim. An analogous generic theory for autonomous Hamiltonian systems, although sketched in its beginnings, was never quite pushed to completion. Adding some artificial dissipation, however, provides some global results; see e.g. [Fiedler] (1988), Section 8.4.2. But for reversible systems no systematic "dissipation trick" is known and a "snakes theory" is still missing entirely. Not even to speak of reversible systems which are in addition equivariant with respect to a linear group action on $x \in \mathbf{R}^{2N}$....

Along a continuous branch of reversible periodic orbits the minimal period may become unbounded in several ways. One possibility is *homoclinic period blow-up*: the periodic orbits approach a reversible homoclinic orbit and disappear in a "blue sky catastrophe." For a recent account of this phenomenon, observed already by [Devaney] (1976), see [Vanderbauwhede & Fiedler] (1991). In a *reversible traveling wave* setting, this effect can be described as a family of spatially periodic traveling waves limiting onto a single pulse wave. In [Chow & Deng & Fiedler] (1990) it was argued that the "snakes" orbit index $\phi(\gamma)$ of approximating periodic orbits γ, mentioned in (3.17) above, can be used for continuation purposes of the limiting homoclinic orbit. For a detailed exposition see [Fiedler] (1992). Whether a similar scheme works in the reversible case, with deg (γ) replacing $\phi(\gamma)$ must fortunately also remain open at this time.

Partial differential equations are still another open topic. For example, consider

$$u_{tt} = \pm \Delta_x u + g(u, u_t) \tag{3.18}$$

where $t \in \mathbf{R}$, x is in a bounded domain Ω, $u = u(t, x)$ satisfies appropriate boundary conditions on $\partial\Omega$, and g is even in u_t. Depending on the sign $+$ or $-$ of $\Delta_x u$, the equation is semilinear hyperbolic or elliptic, respectively. Anyhow $u(-t, x)$, $t \in \mathbf{R}$, is a solution if and only if $u(t, x)$ is. In some elliptic cases equation (3.18) can be reduced to a reversible finite-dimensional system, and our degree method readily applies. See e.g. [Kirchgässner] (1982) for a local and [Mielke] (1990) for a global reduction. At present, we are not able to develop a degree theory, directly, for solutions of the infinite-dimensional problem (3.18) which are periodic in t and reversible.

As a curiosity we remark that, even for (reversible) equilibria, the infinite dimensional part of the Floquet spectrum tends to be hyperbolic, for the elliptic equation, whereas it will be "elliptic" for the hyperbolic equation. So much for nomenclature.

Summarizing, let us apply Theorem 2.1 to our original *Neumann problem*

$$\ddot{u}+g(u,\dot{u})=0, \qquad u\in\mathbf{R}^N, \quad 0\leqslant t\leqslant\mu, \tag{3.19}$$

with g even in $\dot{u}$. If, instead, g is odd in u then analogous results can be obtained for the Dirichlet case. But let us focus on the Neumann problem now. Assume, for simplicity

$$u^Tg(u,\dot{u})<0, \qquad \text{for all} \quad |u|_2>C_1, \dot{u}\perp u. \tag{3.20a}$$

Then $|u|_2\leqslant C_1$ for any solution of the Neumann boundary value problem, $0\leqslant t\leqslant\mu$. Also assume at most linear growth

$$|\dot{u}^Tg(u,\dot{u})|\leqslant C_2(1+|\dot{u}|_2^2), \tag{3.20b}$$

uniformly for all $|u(t)|_2\geqslant C_1$ and all $\dot{u}$. Then in addition $|\dot{u}|_2^2\leqslant e^{2C_2\mu}-1$, for $0\leqslant t\leqslant\mu$. These a priori bounds show that assumption (ii) of Theorem 2.1 holds. Since g is even in $\dot{u}$, the linearization at any reversible equilibrium $u(t)\equiv\xi$, $\dot{u}(t)\equiv 0$ is given in block matrix form by

$$\begin{pmatrix} 0 & I \\ -g_u(\xi,0) & 0 \end{pmatrix}; \tag{3.21}$$

g_u denotes the partial derivative. To guarantee assumption (i) we therefore assume

$$\begin{array}{l}\text{if } g(\xi,0)=0 \text{ then } g_u(\xi,0) \text{ does not possess real eigen-}\\ \text{values } \geqslant 0\text{, except for a simple eigenvalue } \pi^2 \text{ at say } \xi=0.\end{array} \tag{3.22}$$

(Here we recall relation (2.4) between the spectrum of $-g_u$ and that of the block matrix (3.21).) Following our remarks above, we cheerfully ignore assumption (iii) and obtain nonconstant solutions u of our boundary value problem (3.19), for any $\mu=T/2>1$.

We recall that (3.19) is *Hamiltonian*, in addition to being reversible, if $g=g(u)=\nabla_u G(u)$ is the gradient of a scalar function G which does not depend on $\dot{u}$. In particular, our theorems remain valid for such Hamiltonian systems. We sketch the existence proof using variational methods. Let

$$I(u)=\int_0^{T/2}(\dot{u}^2/2-G(u))(t)\,dt \tag{3.23}$$

for $u \in H^1(0, T/2)$. The condition (3.20a) on $g(u)$ implies that $G(u)$ is uniformly bounded from above. Thus $I(u)$ is bounded from below. For $T > 2$ it is easily seen, that $u \equiv 0$ is a saddle for the functional (3.23). This implies the existence of nontrivial minimizers, proving Theorem 2.1 in this case.

For general, not necessarily reversible Hamiltonian systems there is an enormous literature concerning periodic solutions. Mostly, the crucial ingredient is a variational principle on the space of closed loops, with periodic solutions as critical points. See for example [Rabinowitz] (1983, 1986), [Zehnder] (1987), [Ekeland] (1985), [Struwe] (1990), and the references there. One such result, due to [Rabinowitz] (1983), states that for any $T > 0$ there exists a solution of (not necessarily minimal) period T, provided the Hamiltonian grows superquadratically at infinity. Drastically oversimplified, we sketch a corresponding bifurcation in Figure 3.2a. Under additional assumptions, minimality of the period T was also investigated successfully; see e.g. [Ekeland & Hofer] (1985) and the references there. In Figure 3.2b, we similarly oversimplify our Theorem 2.1, for comparison.

Neither result actually obtains continuous branches of periodic solutions. Note that our result does not use any variational structure. The difference in Figs. 3.2a and 3.2b is due to the different conditions at infinity, only, in our opinion.

Conversely it is tempting to extend our *orbit degree* $\deg_n(\gamma)$ to the general Hamiltonian case, even when there is no reversibility. Indeed, we suggest to take Theorem 2.1, (2.4) in [Fiedler & Heinze] (1996) for a *definition* of an orbit degree rather than a consequence. In exchange, homotopy invariance properties then have to be proved. It seems viable to achieve this by a detailed analysis of subharmonic bifurcations for Hamiltonian systems according to the same list as given in Figure 3.1 for the reversible case, see [Meyer] (1970). This time the horizontal axis might indicate period or, alternatively, energy. We find it intriguing, in this context, that the Maslov index of a Hamiltonian periodic orbit, governing an appropriate Morse theory, can likewise be expressed in terms of certain

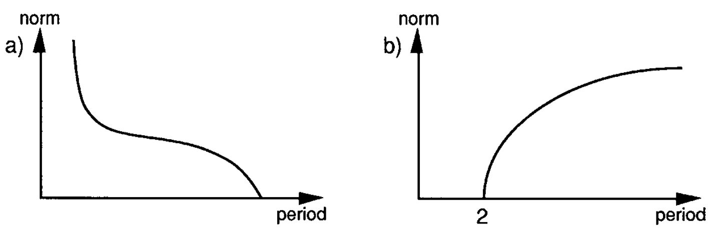

FIG. 3.2. Global branches of periodic orbits. (a) Superquadratic Hamiltonian, (b) a priori bounded reversible case.

Floquet multipliers on the unit circle; see [Zehnder & Salamon] (1988). The periodically forced nonautonomous case, being the main emphasis there, should be seen as a symplectic companion in conjunction with the case of reversible diffeomorphisms. Distinguishing carefully between even and odd iterates, we hope an analogous degree theory for reversible diffeomorphisms can be developed—but not in the present paper.

In absence of elliptic periodic reversible orbits, the solutions in Theorem 2.1 provide injective maps

$$(u, \dot{u})\colon [0, \mu] \to \mathbf{R}^{2N}, \tag{3.24}$$

except for possibly identical endpoints in the case of a Möbius orbit of minimal period $T/2 = \mu$. This injectivity replaces, for systems, monotonicity of u which can be asserted trivially for the case $N=1$. In the case $N=1$, even the *stability* for the corresponding scalar *parabolic equation*

$$u_t = u_{xx} + g(u, u_x), \qquad 0 \leqslant t \leqslant \mu \tag{3.25}$$

can be determined. In fact, (3.19) is satisfied by equilibria u of (3.25). The above monotonicity statement implies that the unstable dimension $i(u)$ satisfies

$$\begin{aligned} i(u) &= 1, \qquad \text{if} \quad \deg_1(\gamma) = 1, \\ i(u) &= 2, \qquad \text{if} \quad \deg_1(\gamma) = -1 \end{aligned} \tag{3.26}$$

where γ denotes the periodic solution of (3.19) of minimal period $T = 2\mu$ which corresponds to the equilibrium u. This claim follows essentially from time map arguments and a Sturm–Liouville comparison with u_x. See [Brunovsky & Fiedler], (1988), Lemma 5.1, [Smoller] (1983), Lemma 24.16, for related arguments.

References

M. Abramowitz and I. A. Stegun (Eds.), "Handbook of Mathematical Functions," Dover, New York, 1965.

J. C. Alexander and J. A. Yorke, Global bifurcation of periodic orbits, *Amer. J. Math.* **100** (1978), 263–292.

P. Brunovsky and B. Fiedler, Connecting orbits in scalar reaction diffusion equations, *in* "Dynamics Reported 1," pp. 57–89, Springer-Verlag, Berlin 1988.

S.-N. Chow, B. Deng, and B. Fiedler, Homoclinic bifurcation at resonant eigenvalues, *J. Dynamics Diff. Eq.* **2** (1990), 177–244.

S.-N. Chow and J. Mallet-Paret, The Fuller index and global Hopf bifurcation, *J. Diff. Eq.* **29** (1978), 66–85.

S.-N. Chow, J. Mallet-Paret, and J. A. Yorke, A periodic orbit index, which is a bifurcation invariant, *in* "Geometric Dynamics" (J. Palis, Jr., Ed.), Lect. Notes in Math., Vol. 1007, Springer-Verlag, New York 1983.

K. Deimling, "Nonlinear Functional Analysis," Springer-Verlag, Berlin 1985.
R. Devaney, Reversible diffeomorphisms and flows, *Trans. Amer. Math. Soc.* **218** (1976), 89–113.
P. G. Lejeune-Dirichlet, "Werke II," G. Reimer, Berlin 1897.
I. Ekeland, "Convexity Methods in Hamiltonian Mechanics," Springer-Verlag, Berlin, 1990.
I. Ekeland and H. Hofer, Periodic solutions with prescribed minimal period for convex autonomous Hamiltonian systems, *Invent. Math.* **81** (1985), 155–188.
B. Fiedler, An index for global Hopf bifurcation in parabolic systems, *J. Reine Angew. Math.* **359** (1985), 1–36.
B. Fiedler, "Global Bifurcation of Periodic Solutions with Symmetry," Lect. Notes in Math., Vol. 1309, Springer-Verlag, Berlin/New York, 1988.
B. Fiedler, Global pathfollowing of homoclinic orbits in two-parameter flows, preprint 1992.
B. Fiedler and S. Heinze, Homotopy invariants of time reversible periodic orbits I: Theory, *J. Diff. Eq.* **126** (1996), 184–203.
K. Geba and W. Marzantowicz, Global bifurcation of periodic solutions, *Topol. Meth. in Nonl. Anal.* **1** (1993), 67–93.
J. Ize, I. Massabo, and V. Vignoli, Degree theory for equivariant maps, *Trans. Amer. Math. Soc.* **315** (1989), 433–510.
K. Kirchgässner, Wave-solutions of reversible systems and applications, *J. Diff. Eq.* **45** (1982), 113–127.
A. Lasota and J. A. Yorke, Bounds for periodic solutions of differential equations in Banach spaces, *J. Diff. Eq.* **10** (1971), 83–91.
J. Mallet-Paret and J. A. Yorke, Snakes: oriented families of periodic orbits, their sources, sinks, and continuation, *J. Diff. Eq.* **43** (1982), 419–450.
K. Meyer, Generic bifurcation of periodic points, *Trans. Amer. Math. Soc.* **149** (1970), 95–107.
A. Mielke, Essential manifolds for elliptic problems in an infinite strip, *J. Diff. Eq.* **110** (1994), 332–355.
G. Polya and G. Szegő, "Problems and Theorems in Analysis," Springer-Verlag, Berlin, 1976.
P. H. Rabinowitz, Periodic solutions of large norm of Hamiltonian systems, *J. Diff. Eq.* **50** (1983), 33–48.
P. H. Rabinowitz, "Minimax Methods in Critical Point Theory with Applications to Differential Equations," Memoirs AMS, Vol. 65, Providence, RI, 1986.
D. Salamon and E. Zehnder, Floer homology, the Maslov index, and periodic orbits of Hamiltonian equations, Univ. Warwick, preprint, 1989.
R. Schaaf, "Global Solution Branches of Two Point Boundary Value Problems," Lect. Notes in Math., Vol. 1458, Springer-Verlag, Heidelberg, 1990.
M. Sevruyk, "Reversible Systems," Lect. Notes Math., Vol. 1211, Springer-Verlag, New York, 1986.
S. Smale, Differentiable dynamical systems, *Bull. AMS* **73** (1967), 747–817.
J. Smoller, "Shock Waves and Reaction Diffusion Equations," Grundlehren d. Math. Wiss., Vol. 258, Springer-Verlag, New York, 1983.
M. Struwe, "Variational Methods," Springer-Verlag, Berlin, 1990.
A. Vanderbauwhede, "Local Bifurcation and Symmetry," Pitman, Boston, 1982.
A. Vanderbauwhede, Subharmonic branching in reversible systems, *SIAM J. Math. Anal.* **21** (1990), 954–979.
A. Vanderbauwhede and B. Fiedler, Homoclinic period blow-up in reversible and conservative systems, *Z. Angew. Math. Phys.* **43** (1992), 292–318.
Z. Zehnder, Periodische Lösungen von Hamiltonschen Systemen, *Jahresber. Deutsch. Math.-Verein.* **89** (1987), 33–59.